# Mathematical Physics

# Team Contest

**To complete this assessment, answer at least 2 of the following 3 problems.**

## Problem 1

### Scattering off a constant magnetic strip

Consider a non-relativistic electron with mass $m$ and charge $e$ in a two-dimensional plane $(x_1, x_2)$. In what follows, you can set $\hbar = c = 1$ for convenience. Now the electron is placed in a perpendicular magnetic field $\hat{B}$ that is constant within an infinite strip and vanishes elsewhere:

$$\hat{B} = \begin{cases} B\hat{x_3}, & x_2 \in [-L, 0] \\ 0, & \text{otherwise} \end{cases} \tag{0.1}$$

where $B > 0$. Recall that:

$$\hat{B} = \vec{\nabla} \times \vec{A} \rightarrow B = \partial_1 A_2 - \partial_2 A_1 \tag{0.2}$$

1. Consider the electron scattering from the asymptotic region $x_2 \rightarrow \infty$ at incident angle $\theta$ along the $x_2$-axis, and with energy $E$. Solve the wave-function of the electron in the asymptotic region $x_2 \rightarrow -\infty$. How large should $E$ be for the electron to scatter through as a plane-wave? In these cases, calculate the out-going angle $\theta'$ of the electron in the asymptotic region $x_2 \rightarrow -\infty$ region.

2. Now we take the limit $L \rightarrow \infty$, so that the magnetic field is present for the full half-plane $x_2 < 0$. In this limit, the system only contains bound-states in the region $x_2 \rightarrow -\infty$. Consider the sector of Hilbert-space characterised by:

   - Having zero momentum along $x_1$ direction: $p_1 = 0$.
   - Satisfying the total reflection condition along $x_2$ direction at $x_2 = 0$, i.e:

   $$\psi(x_1, x_2) \propto \cos{(k_2 x_2)}, \quad x_2 > 0 \tag{0.3}$$

   In this sector, find the energy spectrum.

## Problem 2

### QFT with an external gauge potential

Consider a massless complex scalar field coupled to an external gauge potential

$$S = \int d^4x \left[-g^{\mu\nu}(\partial_\mu + iqA_\mu)\phi(\partial_\nu - iqA_\nu)\phi^*\right] \ , \tag{0.4}$$

where the external gauge potential $A_\mu = (A_0(x), 0, 0, 0)$ (here and hereafter, $x$ denote the first spatial direction, not a four-dimensional vector).

1. Write down the equation of motion for the field operator $\phi$.

2. Consider a solution $\phi_\omega = e^{-i\omega t}\psi_\omega(x)$. Write down the differential equation for $\psi_\omega(x)$.

3. Consider a step potential

$$qA_0(x) = \begin{cases} 0 & (x < 0) \\ \mu & (x > 0) \end{cases} \quad . \tag{0.5}$$

   We will only consider $\phi_\omega$ where $\omega < \mu$.

   When $t \to -\infty$, consider an incoming normalizable "wave packet" (i.e., one can integrate over a range of $\omega$ to localize the wave packet, and the motion of the wave packet is characterized by its group velocity) of the field operator $\phi_\omega = c_\omega e^{-i\omega t + i\omega x}$ which is located at $x \to -\infty$. Find the solution of $\phi_\omega$ in the limit $t \to +\infty$.

4. At $x \to \pm\infty$, we expect that the quantization of our model should be the same as that of free fields:

$$\phi(t,x,y,z) = \begin{cases} \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2|\mathbf{k}|}}\left(a_{\mathbf{k}}e^{-i|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{x}} + b_{\mathbf{k}}^\dagger e^{i|\mathbf{k}|t-i\mathbf{k}\cdot\mathbf{x}}\right) & (x \to -\infty) \\ e^{-i\mu t}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2|\mathbf{k}|}}\left(\tilde{a}_{\mathbf{k}}e^{-i|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{x}} + \tilde{b}_{\mathbf{k}}^\dagger e^{i|\mathbf{k}|t-i\mathbf{k}\cdot\mathbf{x}}\right) & (x \to +\infty) \end{cases} \tag{0.6}$$

   where $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are the annihilation operators for particle and anti-particle at $x \to -\infty$, respectively. $\tilde{a}_{\mathbf{k}}$ and $\tilde{b}_{\mathbf{k}}$ are the corresponding quantities at $x \to +\infty$.

   Compare our $\phi_\omega$ and $\phi(t,x,y,z)$, we get, up to a normalization factor, at $t \to -\infty$, $x \to -\infty$:

$$\phi_\omega \propto a_{(\omega,0,0)}e^{-i\omega t + i\omega x} \ , \tag{0.7}$$

   where $(\omega, 0, 0)$ is a momentum vector along the $x$ direction. In other words, $\phi_\omega$ plays the role of the annihilation operator at the initial time.

   At $t \to +\infty$, compared with the quantization of a free field at $x \to \pm\infty$, write down $\phi_\omega$ in terms of the creation and annihilation operators (up to the same normalization factor as in (0.7)), and describe the physical meaning of $\phi_\omega$ at $t \to +\infty$.

5. If we set up our initial state to be the vacuum state where $\langle \phi_\omega^\dagger \phi_\omega \rangle_{\text{in}} = 0$, then qualitatively describe what happens afterwards.

## Problem 3

### General Relativity

1. **Conservation laws**: Let $(M, g_{ab})$ be a 4-dimensional vacuum spacetime. Recall that a Killing-Yano 2-tensor is a 2-form $Y_{ab} = Y_{[ab]}$, satisfying $\nabla_{(a}Y_{b)c} = 0$.

   (a) Show that $\nabla_a Y_{bc} = \nabla_{[a}Y_{bc]}$

   (b) Use the result from (a) to show that for $S_1, S_2$ closed surfaces in $M$ bounding a 3-volume $\Sigma$, it holds that

$$\int_{S_1} R^{abcd}Y_{cd}dS_{ab} = \int_{S_2} R^{abcd}Y_{cd}dS_{ab}$$

2. **Stress-energy tensor**: Show that the Maxwell action

$$\mathcal{S} = \frac{1}{16\pi} \int F_{ab} F^{ab} \sqrt{|\det g|} d^4 x$$

is conformally invariant and use this to show the Maxwell stress-energy tensor is traceless.

3. **Particle horizon**: Let $M = (0, \infty) \times \Sigma$ where $\Sigma$ is hyperbolic 3-space. Assume that $M$ has line element

$$ds^2 = -dt^2 + t^2 \gamma$$

for $t > 0$. Let $P, Q$ be two events at time $t_0 > 0$. Show that there is an event to the past of $P, Q$ that can send signals to both $P, Q$. Is the same true for the line element $-dt^2 + t^{3/2}\gamma$?.